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Replica symmetry breaking in the minority game

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Abstract

We extend and complete recent work concerning the analytic solution of the minority game. Nash equilibria (NE) of the game have been found to be related to the ground states of a disordered Hamiltonian with replica symmetry breaking (RSB), signalling the presence of a large number of NE. Here we study the number of NE both analytically and numerically. We then analyse the stability of the recently obtained replica-symmetric solution and, in the region where it becomes unstable, derive the solution within one-step RSB approximation. We are finally able to draw a detailed phase diagram of the model.

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1. Introduction

The minority game (MG) has drawn much attention recently as a toy model of a market [1,2][1]. In the simplest possible case, when no public information [3] is present, its definition is fairly simple. At each time step, N players have to choose between two actions, such as buying a certain stock or selling it. Those who end up on the minority side win. This mechanism can be obtained by abstracting the well known law of supply-and-demand. When the majority of traders are buying a certain asset it is convenient to be a seller, for prices are likely to be high, and vice versa. The minority side has an advantage.

The full complexity of the model arises in the presence of public information, which is modelled by the occurrence of one of P events representing, e.g., some political news or a price change. In the minority game agents resort to choice rules or information-processing devices—called strategies henceforth—which suggest whether to 'buy' or 'sell' depending on the information they have received. Then each player acts according to the suggestion of his best performing strategy. This mechanism allows one to tackle the central problem one faces when trying to understand the collective behaviour of systems of heterogeneous agents interacting under strategic interdependence (as in markets), that is, how agents react to public information (e.g., political news or price changes) and the feedback effects that these reactions have on public information.

[1] See the webpage www.unifr.ch/econophysics/minority for history and a complete list of references.

Early numerical simulations [1, 4–6] have shown a remarkably rich behaviour where both cooperativity and crowd effects [6, 7] arise. Much emphasis was initially put on the emergence of a cooperative phase in the stationary state as compared to the reference situation of random agents—i.e., agents who toss a coin to decide which action to take. Agents in the minority game are able to coordinate their actions and reduce the global waste of resources below the level corresponding to random agents.

Later work [8–11] has revealed that the agents' adaptive dynamics minimizes a global function, related to market predictability, and that the system undergoes phase transition between an asymmetric and a symmetric, unpredictable market, as the ratio $\alpha = P/N$ decreases. Full characterization of the model's behaviour for $N \to \infty$ was derived studying the minima of the global function by the replica method [9–11][2]. It was also realized that agents can greatly improve their performance and global efficiency if they account for their own impact on the market [3, 9, 10]. In this case the steady state is a *Nash equilibrium*, that is a configuration where no agent can improve his performance by changing his behaviour if others stick to theirs. Also in this case the dynamics is related asymptotically to the minima of a global function, and hence statistical mechanics again allows one to describe in detail the stationary state. However, for Nash equilibria (NE) replica symmetry breaking (RSB) occurs. This makes the replica symmetric (RS) calculation of [9, 10] only an approximation.

In this paper we take the first steps towards a complete characterisation of the set of NE of the minority game. Our analysis will be slightly more general, for we shall alsobe able to embody the cooperative state. First, we shall briefly outline the replica approach, showing that the RS solution has a limited validity due to entropy arguments. Then we compute the number of NE, following [12], and show that there are exponentially many (in N). By analogy with the de Almeida–Thouless (AT) stability analysis of the SK spin glass model [13, 14], we find the phase transition line (AT line) separating the RS from the RSB phase. Finally we shall break the RS and study the solution in the one-step RSB (1RSB) approximation. The exact solution probably requires infinitely many RSB steps but 1RSB does provide an extremely close agreement with numerical results. Finally, we will be able to draw a complete phase diagram of the model within 1RSB.

Our discussion will focus on the statistical mechanical properties of the model. The economic and game-theoretic aspects of the model, which are discussed in detail elsewhere [2, 3, 10, 11], will only be described briefly.

2. The model

2.1. Basic definitions

The essential ingredients of the minority game are:

- N players, labelled by the index i;
- for each player i a strategy variable $s_i \in \{\pm 1\}$, saying which strategy (+1 or -1) player i is adopting (we restrict ourselves to the case where players have two strategies each);
- P different information patterns, labelled by the index μ.

At each time step t all players receive the same information μ, drawn at random with equal probability in $\{1, \ldots, P\}$ [15]. Strategies s are the labels of information processing devices, that suggest an action as a value of a binary variable (like 'buy' or 'sell') upon receiving

[2] It should be mentioned at this point that the replica approach fails to describe the system's behaviour in a certain range of the parameters, as discussed in [3, 17]. This point will be made more precise later in the text.

information μ

$$s : \{1, \ldots, P\} \ni \mu \mapsto a_{i,s}^{\mu} \in \{\pm 1\}. \tag{1}$$

Two such strategies are assigned to each agent and they are drawn at random and independently for each agent, from the set of all 2^P such functions. In practice, $a_{i,s}^{\mu}$ plays the role of quenched disorder, analogous to the random couplings $\{J_{ij}\}$ in spin glass models.

It is convenient to make the dependence of $a_{i,s}^{\mu}$ on s explicit by introducing the auxiliary random variables ω_i^{μ} and ξ_i^{μ} such that

$$a_{i,s}^{\mu} = \omega_i^{\mu} + s\xi_i^{\mu}. \tag{2}$$

Clearly, both ξ_i^{μ} and ω_i^{μ} take on values in $\{0, \pm 1\}$ but they are not independent.

The payoff to player i under information μ is defined as

$$u_i^{\mu}(s_i, s_{-i}) = -a_{i,s_i}^{\mu} A^{\mu} \qquad A^{\mu} = \sum_{j=1,N} a_{j,s_j}^{\mu} \tag{3}$$

where $s_{-i} = \{s_j\}_{j \neq i}$. It is positive whenever i is in the minority group, hence the name of the game. Moreover, players interact with each other only through a global quantity (namely, A^{μ}). This feature clarifies the mean-field character of the model. The total loss experienced by players under information μ simply reads

$$-\sum_{i=1,N} u_i^{\mu}(s_i, s_{-i}) = (A^{\mu})^2 \tag{4}$$

which is always positive.

2.2. Dynamics

A snapshot configuration of the system corresponds to a point $\{s_i\}_{i=1}^N$ in the (pure) *strategy space* $\{\pm 1\}^N$. The game is repeated and at each time step players face the problem of choosing the strategy to follow. By assumption, each player keeps a 'score' $U_{i,s}(t)$ for each strategy $s = \pm 1$ and updates it as the game proceeds. In the beginning, players set $U_{i,\pm 1}(0) = 0$. Then for $t \geqslant 0$ scores are updated according to the map

$$U_{i,s}(t+1) = U_{i,s}(t) - \frac{1}{P} a_{i,s}^{\mu(t)} [A^{\mu(t)} - \eta(a_{i,s_i(t)}^{\mu(t)} - a_{i,s}^{\mu(t)})] \tag{5}$$

where $\eta \in \mathbb{R}$ and $s_i(t)$ denotes the strategy that player i actually uses at time t. The term proportional to η is introduced to model agents who account for their market impact. We refer the reader to [10] for a detailed discussion of this term[3].

Let it be suffice to say that with $\eta = 0$ equation (5) reduces to the standard minority game dynamics: in this case agents reward (penalize) strategies which would have prescribed a sign opposite (equal) to that of A^{μ}. In doing so, agents ignore the fact that if they actually had played those strategies, their contribution to A^{μ}, and hence A^{μ} itself, could have changed. With $\eta = 1$ instead, agents correctly account for their contribution to A^{μ}. Hence the reward to strategy s is really the payoff that agent i would have received had he played that strategy. The parameter η tunes the extent to which agents account for their market impact.

Following [16], we assume that the probability with which player i chooses the strategy to adopt at time t depends on the strategy's score as follows:

$$\mathrm{Prob}\{s_i(t) = \pm 1\} = C \exp[\Gamma U_{i,\pm 1}(t)] \tag{6}$$

[3] In [3, 9, 10] the term proportional to η is $\eta\delta_{s,s_i(t)}/P$. This leads, however, to the same results of the last term in equation (5) in the statistical mechanics approach. The reason for this is that the approach of [9, 10] is based on the average of the evolution equation in the stationary state. Observing that $a_{i,s_i(t)}^{\mu(t)} a_{i,s}^{\mu(t)}$ is 1 when $s_i(t) = s$ and a random sign, with zero average otherwise, we find that the time average of the last term of equation (5) is the same as the average of $\eta(\delta_{s,s_i(t)} - 1)/P$. Hence the two equations are equivalent (apart from an irrelevant constant $-\eta/P$).

where C is a normalization constant and $\Gamma > 0$ is the learning rate. With this rule, the most successful strategy is more likely to be chosen[4].

Note that A^μ is the contribution of N terms whereas the term proportional to η in equation (5) is of order 1. One may naively argue that the η term is irrelevant as $N \to \infty$. This is not so for exactly the same reason for which the Onsager reaction term—or cavity field—is relevant in mean-field spin glass theory [13][5]. Indeed [9, 10] have shown that for $\eta = 1$ the dynamics converges to a NE. This means that, in a sense, players have become fully sophisticated. For $\eta = 0$, instead, agents converge to a sub-optimal state.

We introduce the continuous variables ('soft spins') $\phi_i \in [-1, 1]$ as

$$\phi_i = \langle s_i \rangle \tag{7}$$

where $\langle \cdots \rangle$ stands for the average over the distribution of s_i in the stationary state. The system is then described by a point $\{\phi_i\}_{i=1}^N$ in the hypercube $[-1, 1]^N$. Hence the ϕ_i are the relevant *dynamical variables* and the phase space is $[-1, 1]^N$.

The analytic study of the dynamics equation (5) was carried out in [10]. We shall therefore omit the details and limit ourselves to a brief outline of the results. One can show that the stationary states of the dynamics correspond to the minima of the function

$$H_\eta = \sum_{i \neq j}^{1,N} \overline{\xi_i^\mu \xi_j^\mu} \phi_i \phi_j + 2 \sum_{i=1}^{N} \overline{\Omega^\mu \xi_i^\mu} \phi_i + \eta \sum_{i=1}^{N} \overline{(\xi_i^\mu)^2} (1 - \phi_i^2) + \overline{(\Omega^\mu)^2} \tag{8}$$

where $\overline{\cdots} = (1/P) \sum_{\mu=1,P} \cdots$ and $\Omega^\mu = \sum_{i=1,N} \omega_i^\mu$.

Analysing the stationary states of the dynamics is then equivalent to minimizing H_η. The limiting cases $\eta = 0$ and 1 correspond to

$$H_0 = \overline{\langle A^\mu \rangle^2} \qquad \text{and} \qquad H_1 = \overline{\langle (A^\mu)^2 \rangle}, \tag{9}$$

respectively. H_0 (whose minima describe the standard minority game) is related to the market's predictability or available information, as explained at length in [8–10]. In fact, if $\langle A^\mu \rangle \neq 0$, then one can predict that the action $a^\mu = -\text{sign}\, \langle A^\mu \rangle$ is more likely to be successful than the other whenever pattern μ arises. H_1 is the long time total loss of players averaged over μ, as is clear from equation (4). In previous works this quantity is usually denoted as σ^2. We stress the fact that the minima of H_1 are the game's NE.

It is easy to understand that [9, 10]:

(1) H_0 is a positive definite quadratic form. Hence for $\eta = 0$ there is a unique stationary state, corresponding to the cooperative state observed in early numerical simulations;
(2) for any $\eta > 0$ both the global efficiency and the individual payoffs are sensibly improved with respect to the $\eta = 0$ case;
(3) for $\eta = 1$ there is a large number of stationary states, i.e., of NE. These states have $\phi_i^2 = 1$, i.e., agents play pure strategies.

Point (1) has been treated extensively in previous works. Here we shall focus on points (2) and (3), namely on the $\eta > 0$ case.

[4] For $\eta = 0$, our approach gives correct results only for sufficiently large Γ, namely for $\Gamma > \Gamma_c(\alpha)$ of [3].

[5] While the contribution of other spins to the effective field acting on spin i have fluctuating signs, the contribution of spin i always has the sign of spin i. Mean-field theory needs to be corrected with the subtraction of the self-interaction from the effective-field, which becomes a cavity field. Likewise the contribution of agent $j \neq i$ to A^μ is uncorrelated with the action of agent i, whereas the contribution of agent i itself is totally correlated with his action.

3. Replica approach

3.1. Replica-symmetric theory

In order to minimize the function in equation (8) we can resort to statistical mechanics methods, for

$$\min_{\{\phi_i\}_{i=1}^N \in[-1,+1]^N} H_\eta = -\lim_{\beta\to\infty} \frac{1}{\beta} \log Z(\beta) \equiv \lim_{\beta\to\infty} F_\eta(\beta) \tag{10}$$

where $Z(\beta)$ is the canonical partition function associated with H_η. Further, since H_η contains quenched disorder we need to apply the replica formalism [13] to analyse its ground states. If we let $J = \{a^\mu_{i,s_i}\}$ denote collectively the disorder variables and $E_J(\cdots)$ denote the statistical average over J, the 'typical free energy' $F_\eta(\beta)$ can be obtained from the identity

$$E_J[\log Z(\beta)] = \lim_{n\to 0} \frac{E_J[Z^n(\beta)] - 1}{n}. \tag{11}$$

A long but standard computation (see appendices in [10,11]) leads to the following expression:

$$\begin{aligned} F_\eta(\beta) = {} & \frac{\alpha}{2\beta n} \operatorname{Tr}\left\{\log\left[\left(1+\frac{\beta}{\alpha}\right)\mathbb{I}_n + \frac{\beta}{\alpha}\hat{q}\right]\right\} + \frac{\alpha\beta}{2n}\sum_{a\neq b}^{1,n} r_{ab}q_{ab} \\ & -\frac{1}{\beta n}\log\left\{\operatorname{Tr}_s\left[\exp\left(\frac{\alpha\beta^2}{2}\sum_{a\neq b}^{1,n} r_{ab}q_{ab}\right)\right]\right\} + \frac{\eta}{2}(1-Q_a) \end{aligned} \tag{12}$$

where $\mathbb{I}_n$ is the n-dimensional unit matrix, $\hat{q}$ is the *overlap matrix* with elements $q_{ab} = \langle\phi_i^a\phi_i^b\rangle$ $(a, b = 1, \ldots, n;\ a \neq b)$, and $Q_a = (1/N)\sum_{i=1,N}(\phi_i^a)^2$ $(a = 1, \ldots, n)$. The quantities r_{ab} and R_a appear as Lagrange multipliers associated to q_{ab} and Q_a, respectively.

Imposing the RS ansatz ($q_{ab} = q$ for all $a \neq b$, $Q_a = Q$ for all a, and similarly for r_{ab} and R_a) one obtains

$$\begin{aligned} F_\eta^{(\mathrm{RS})}(\beta) = {} & \frac{\alpha}{2}\frac{1+q}{\alpha+\beta(Q-q)} + \frac{\alpha}{2\beta}\log\left[1+\frac{\beta(Q-q)}{\alpha}\right] + \frac{\beta}{2}(RQ - rq) \\ & -\frac{1}{\beta}\left\langle \log\int_{-1}^{1}\exp[-\beta V(s;z)]\,\mathrm{d}s\right\rangle_z + \frac{\eta}{2}(1-Q) \end{aligned} \tag{13}$$

with

$$V(s;z) = -\sqrt{\frac{\alpha r}{2}}zs - \frac{\alpha\beta}{4}(R - r_1)s^2. \tag{14}$$

The ground state properties of H_η in the $N \to \infty$ limit can now be studied by solving the saddle-point equations (obtained by setting equal to zero the derivatives of $F_\eta^{(\mathrm{RS})}(\beta)$ with respect to Q, q, R and r) in the $\beta \to \infty$ limit, since

$$\lim_{N\to\infty} \min_{\{\phi_i\}_{i=1}^N\in[-1,+1]^N} \frac{H_\eta}{N} = \lim_{\beta\to\infty} \frac{F_\eta^{(\mathrm{RS})}(\beta)|_{\mathrm{sp}}}{N} \tag{15}$$

the subscript 'sp' indicates that the function $F_\eta^{(\mathrm{RS})}(\beta)$ was computed at the saddle-point values of Q, q, R and r. This procedure yields the so-called RS solution.

For $\eta = 0$ this solution is characterized by a phase transition at $\alpha_c \simeq 0.3374\ldots$ separating a symmetric phase ($\alpha < \alpha_c$) with $H_0 = 0$ from an asymmetric one ($\alpha > \alpha_c$) with $H_0 > 0$. The 'spin susceptibility' $\chi = \beta(Q - q)/\alpha$ diverges as $\alpha \to \alpha_c^+$. Also, it is known that this solution is stable against RSB for all values of the control parameter α. A detailed account of this case can be found in [9].

We expect the RS to breakdown for all $\eta > 0$ at certain critical values of α, denoted by $\alpha_{\mathrm{AT}}(\eta)$. In order to test this prediction, we study the stability of the RS solution for generic η.

We shall see that for the $\beta \to \infty$ limit a RSB phase arises. It is natural then to ask whether there is a critical temperature β_{c} separating a high temperature behaviour from a low temperature one. This question, even if not directly related to the minority game, may be of interest on its own and can be answered at the RS level. Without going into details, let it be suffice to mention that by setting $q = 0$ in the RS saddle-point equations—which is correct for the high temperature phase—one finds that $\beta_{\mathrm{c}} = 0$ for all values of α and η.

3.2. Entropy of the TS solution for $\eta = 1$

In random Ising spin systems useful indications about the stability of the RS solution are provided by zero-temperature entropy, namely

$$S_{\eta}^{(\mathrm{RS})}(\beta \to \infty) = \lim_{\beta\to\infty} \beta^{2} \frac{\partial F_{\eta}^{(\mathrm{RS})}(\beta)}{\partial \beta^{2}}. \tag{16}$$

This quantity is non-negative due to the discreteness of the configuration space of the model (i.e., $\{\pm 1\}^{N}$). If its zero-temperature limit turns out to be negative, the corresponding solution is unstable and further steps in the approximation are needed.

In our case, this point is more subtle than it seems. In fact, H_{η} is defined for continuous variables $\phi_i \in [-1, +1]$, and not for Ising (i.e., discrete) variables. The corresponding configuration space is not a discrete set of points, but rather a continuum. Therefore, in principle, the zero-temperature entropy need not be non-negative.

For $\eta = 1$, however, H_{η} attains its minima on the corners of the phase space[6] $[-1, 1]^{N}$, i.e., $\phi_i = \pm 1$ for all i. Hence the zero-temperature entropy calculation is revealing. Leaving details of the computation aside, the final result is that for $\alpha > 1/\pi$

$$S_{1}^{(\mathrm{RS})}(\beta \to \infty) = \frac{\alpha}{2}\left[\frac{C}{\alpha + C} - \log\left(1 + \frac{C}{\alpha}\right)\right] \qquad C = \frac{\alpha}{\sqrt{\pi\alpha} - 1}. \tag{17}$$

This is negative for all $\alpha > 1/\pi$ and it diverges at $\alpha = 1/\pi$. For $\alpha < 1/\pi$ one finds $S_1 = -\infty$. This means that RSB occurs for all values of α at $\eta = 1$, or, in other words, that the study of NE requires RSB.

3.3. The number of NE

A crucial feature of the occurrence of RSB is gained from the study of the number of minima of H_1. We show here that an exponentially (in N) large number of such minima occurs, i.e., that the game possesses an exponentially large number of NE. Our analytic results will be supported by numerical investigations.

In order to compute the number of NE we use the fact that NE are pure strategies, or, that at any NE $\phi_i = \pm 1$ for all i. Keeping this in mind, we start by considering that NE satisfy the condition

$$\phi_i\left[\overline{u_i^{\mu}(+1, s_{-i})} - \overline{u_i^{\mu}(-1, s_{-i})}\right] = 2\left[\overline{(\xi_i^{\mu})^2} - \overline{A^{\mu}\xi_i^{\mu}}\phi_i\right] \geqslant 0 \qquad \forall i. \tag{18}$$

Hence an indicator function for NE (using $2\overline{(\xi_i^{\mu})^2} \simeq 1$) is

$$I_{\mathrm{NE}}(\{\phi_i\}) = \prod_{i=1}^{N} \theta\left(1 - 2\overline{A^{\mu}\xi_i^{\mu}}\phi_i\right) \tag{19}$$

[6] This is a consequence of the fact that H_1 is an harmonic function of ϕ_i, i.e. $\nabla_{\phi}^{2} H_1 = 0$. This implies that extrema occurs on the corners.

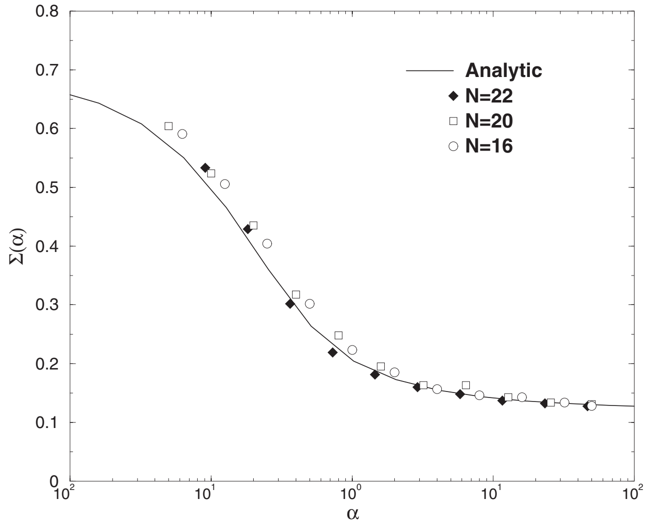

Figure 1. Logarithm of the average number of NE divided by N (Σ) as a function of α.

($I_{\rm NE} = 1$ if $\{\phi_i\}$ is a NE and $= 0$ otherwise) and the number of NE is just obtained summing over all configuration $\{\phi_i\} \in \{\pm 1\}^N$ (an operation which we denote by Tr_ϕ). Hence $\mathcal{N}_{\rm NE} = \mathrm{Tr}_\phi\, I_{\rm NE}\{\phi_i\}$. Following [12] we take the average over the disorder and introduce the integral representation of the θ function. We arrive at[7]

$$E_J(\mathcal{N}_{\rm NE}) = \frac{N^2\alpha^3}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{d}\gamma\,\mathrm{d}\Gamma\,\mathrm{d}\omega\,\mathrm{d}\Omega\,\exp\left[N\Sigma(\gamma,\Gamma,\omega,\Omega)\right] \tag{20}$$

with

$$\Sigma(\alpha,\gamma,\Gamma,\omega,\Omega) = \alpha\omega\gamma + \alpha^2\Omega\Gamma - \frac{\alpha}{2}\log[(1+\gamma)^2 + 2\Gamma] + \log\left[1 + \mathrm{erf}\left(\frac{1-\omega}{2\sqrt{\Omega}}\right)\right] \tag{21}$$

equation (20) is dominated by the saddle-point of ϕ, which is attained at $\omega = 1-\gamma$, $\Omega = \frac{1-\gamma}{\alpha(1+\gamma)}$ and $\Gamma = \frac{\gamma^2(1+\gamma)}{2(1-\gamma)}$, where γ is the root of the equation

$$\frac{\gamma^2}{4}\frac{\alpha(1+\gamma)}{1-\gamma} = \log\left(\gamma\sqrt{\frac{\alpha(1+\gamma)}{1-\gamma}}\right) - \log\left\{\alpha\gamma^2\sqrt{\pi}\left[1 + \mathrm{erf}\left(\frac{\gamma}{2}\sqrt{\frac{\alpha(1+\gamma)}{1-\gamma}}\right)\right]\right\}. \tag{22}$$

In terms of the solution γ^* of this equation we have

$$\Sigma(\alpha) = \frac{\alpha\gamma^*}{2}(2-\gamma^*) - \frac{\alpha}{2}\log\left(\frac{1+\gamma^*}{1-\gamma^*}\right) + \log\left[1 + \mathrm{erf}\left(\frac{\gamma^*}{2}\sqrt{\frac{\alpha(1+\gamma^*)}{1-\gamma^*}}\right)\right]. \tag{23}$$

The behaviour of Σ as a function of α is shown in figure 1. As expected (see [3]), as $\alpha \to 0$ the number of NE grows as 2^N. Numerical results from the exact enumeration of $N \leqslant 20$ are in very good agreement, which shows that the so-called annealed approximation used here (i.e., taking the average of $\mathcal{N}_{\rm NE}$) is sufficient and one does not need to introduce replicas (to compute the average of $\log\mathcal{N}_{\rm NE}$) at this level.

[7] The reader is warned that the Ω appearing here is no relation to the Ω^μ introduced in section 2.

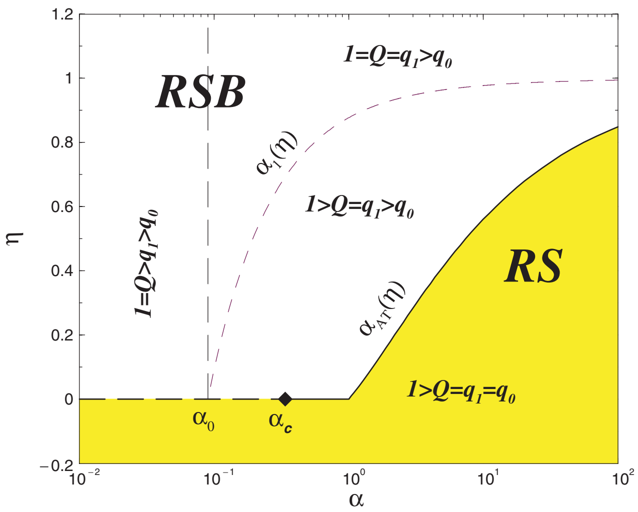

Figure 2. Phase diagram of the minority game, where the shaded region corresponds to the RS phase and the light region to the RSB phase. The AT line marking the boundary between the RS and RSB phases is denoted by $\alpha_{\rm AT}(\eta)$. The point $\alpha_{\rm c} \simeq 0.3374\ldots$ separates a line of second-order transitions (full heavy curve for $\alpha > \alpha_{\rm c}$) from a first-order line of critical points (heavy dashed line for $\alpha < \alpha_{\rm c}$). The two light dashed curves refer to the different regimes found in the solution of the one-step approximation in the RSB phase (see text for details).

3.4. The AT line

A more thorough analysis can be obtained using the AT protocol [14]. In order to investigate the stability of the RS ground states of H_η against RSB we compute the matrix of the second derivatives of the general expression for the free energy, equation (12), with respect to q_{ab} and r_{ab}. The conditions for RSB are then obtained by studying the effect of fluctuations in the RSB direction. This analysis results in an instability line—i.e. a family of points where the RS solution becomes unstable—in the parameter space (α, η), called the AT line and denoted by $\alpha_{\rm AT}(\eta)$. The resulting equation has to be solved (numerically) together with the RS saddle-point equations.

An outline of the calculation is reported in the appendix. We have studied particularly the so-called replicon mode, namely those eigenvectors of the stability matrix which are symmetric under the interchange of all but two of the indices. The replicon mode is typically responsible for the onset of the RSB instability. Points on the AT line are found to satisfy the following stability condition:

$$\alpha\left[1-\eta\left(1+\frac{\beta(Q-q)}{\alpha}\right)\right]^2 = 1. \tag{24}$$

The 'susceptibility' $\chi \equiv \beta(Q-q)/\alpha$ remains finite as $\beta \to \infty$, so that the zero-temperature behaviour can be safely detected. The results are reported in figure 2. For $\eta = 0$ RS is preserved for all α. The point $\alpha_{\rm c} = 0.3374\ldots$ where the second-order phase transition occurs in the standard MG, separates a line of first-order phase transitions, for $\alpha < \alpha_{\rm c}$, from a second-order line. For $\eta = 0^+$ one finds RSB for $\alpha_{\rm AT} = 1^+$. Finally, for $\eta = 1$ RS is broken for all α.

3.5. RSB

The one-step breaking of replica permutation symmetry is expressed by the Parisi ansatz for the q_{ab}s and the Q_as, where an additional parameter denoted by m is introduced: $Q_a = Q$ (all a), $q_{ab} = q_1$ (all $a \neq b$ such that $|a-b| \leqslant m$) and $q_{ab} = q_0$ (otherwise). The 'free energy' is the same as in equation (12), but this time the overlap matrix has to be parameterized as

$$\hat{q} = q_0 \epsilon_n \epsilon_n^T + (q_1 - q_0)\mathbb{I}_{\frac{n}{m}} \otimes \epsilon_m \epsilon_m^T + (Q - q_1)\mathbb{I}_n \tag{25}$$

where ϵ_n is the n-dimensional column vector with all components equal to one (so that $\epsilon_n \epsilon_n^T$ is the n-dimensional matrix with all elements equal to one) and we have used the standard tensor product.

We need to consider the matrix

$$\hat{T} = \left(1 + \frac{\beta}{\alpha}\right)\mathbb{I}_n + \frac{\beta}{\alpha}\hat{q} \tag{26}$$

that is

$$\hat{T} = \left[1 + \frac{\beta}{\alpha}(Q - q_1)\right]\mathbb{I}_n + \frac{\beta}{\alpha}(1 + q_0)\epsilon_n \epsilon_n^T + \frac{\beta}{\alpha}(q_1 - q_0)\mathbb{I}_{\frac{n}{m}} \otimes \epsilon_m \epsilon_m^T. \tag{27}$$

Using the identities

$$\begin{aligned} &\mathbb{I}_n \equiv \mathbb{I}_{\frac{n}{m}} \otimes \mathbb{I}_m \\ &\epsilon_n \epsilon_n^T \equiv \epsilon_{\frac{n}{m}} \epsilon_{\frac{n}{m}}^T \otimes \epsilon_m \epsilon_m^T \end{aligned} \tag{28}$$

we can decompose $\hat{T}$ into tensor products and write its determinant $|\hat{T}|$ in a straightforward way (we need $|\hat{T}|$ because $\mathrm{Tr}(\log \hat{T}) = \log |\hat{T}|$). We get

$$\begin{aligned} |\hat{T}| = &\left[1 + \frac{\beta}{\alpha}(Q - q_1) + n\frac{\beta}{\alpha}(1 + q_0) + m\frac{\beta}{\alpha}(q_1 - q_0)\right] \\ &\times \left[1 + \frac{\beta}{\alpha}(Q - q_1) + m\frac{\beta}{\alpha}(q_1 - q_0)\right]^{\frac{n}{m}-1} \left[1 + \frac{\beta}{\alpha}(Q - q_1)\right]^{n-\frac{n}{m}}. \end{aligned}$$

This means that the first factor on the rhs is an eigenvalue of $\hat{T}$ with multiplicity 1, the second one is an eigenvalue with multiplicity $\frac{n}{m} - 1$, and the third one is an eigenvalue with multiplicity $n - \frac{n}{m}$.

Putting the latter formula into equation (12) and taking the $n \to 0$ limit as requested by the replica trick, we obtain the 1RSB free energy $F_\eta^{(1\mathrm{RSB})}(\beta)$, whose final expression is

$$\begin{aligned} F_\eta^{(1\mathrm{RSB})}(\beta) = &\frac{\alpha}{2}\frac{1 + q_0}{\alpha + \beta(Q - q_1) + m\beta(q_1 - q_0)} + \frac{\eta}{2}(1 - Q) + \frac{\alpha}{2\beta}\log\left[1 + \frac{m\beta(q_1 - q_0)}{\alpha + \beta(Q - q_1)}\right] \\ &+ \frac{\alpha}{2\beta m}\log\left[1 + \frac{m\beta(q_1 - q_0)}{\alpha + \beta(Q - q_1)}\right] + \frac{\alpha\beta}{4}[RQ + (m-1)r_1 q_1 - m r_0 q_0] \\ &- \frac{1}{m\beta}\left\langle \log\left\langle\left(\int_{-1}^{1} \mathrm{d}s \exp[-\beta V(s; y, z)]\right)^m\right\rangle_y\right\rangle_z \end{aligned} \tag{29}$$

where $\langle\cdots\rangle_x$ denotes again the average over the unit Gaussian variables x and

$$V(s; y, z) = -\sqrt{\frac{\alpha r_0}{2}}zs - \sqrt{\frac{\alpha(r_1 - r_0)}{2}}ys - \frac{\alpha\beta}{4}(R - r_1)s^2. \tag{30}$$

$F_\eta^{(1\mathrm{RSB})}(\beta)$ depends on seven parameters: the three overlap matrix elements Q, q_0, q_1, their related Lagrange multipliers R, r_0 and r_1, and m. Their values have to be determined self-consistently from the seven saddle-point equations obtained by setting to zero the derivatives

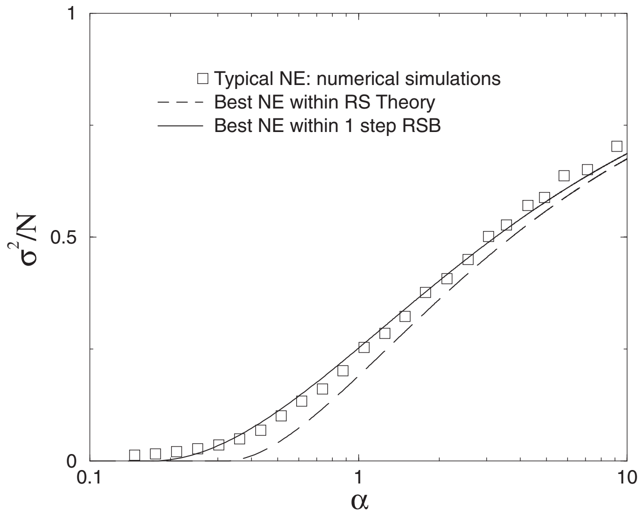

Figure 3. Behaviour of $\sigma^2/N \equiv H_1/N$, corresponding to NE. The one-step solution is compared to numerical data and to the RS solution (dashed curve).

of the free energy with respect to the above parameters. These equations can be solved numerically.

One finds three different regimes in the (α, η) plane when $\beta \to \infty$ (figure 2):

(1) For $\alpha < \alpha_0 \simeq 0.090\,12\ldots$ (all $\eta > 0$) one has $H_\eta = 0$. The solution does not depend on η as long as $\eta > 0$. The self-overlap is $Q = 1$ signalling that agents play pure strategies ($\phi_i = \pm 1$) but off diagonal overlaps $q_1 > q_0$ are both <1. This suggests that NE are organized in a complex geometric structure. The parameter m attains a finite value.

(2) For $\alpha_0 < \alpha < \alpha_1(\eta)$ (all $\eta > 0$) the solution has $H_\eta > 0$, it is independent of η (for $\eta > 0$) and $1 = Q = q_1 > q_0$. The spin susceptibility $\chi = \beta(Q - q_1)/\alpha$ attains a finite value in the limit $\beta \to \infty$, which diverges as $\alpha \to \alpha_0$. Agents again play pure strategies and $q_0 < 1$ is the typical overlap between two NE. The parameter m vanishes as $1/\beta$ (indeed the βm is finite as $\beta \to \infty$). The line $\alpha_1(\eta)$ is determined by the solution of

$$\frac{\eta}{2} = \frac{1}{\alpha + \beta(Q - q_1)}. \tag{31}$$

(3) In-between the line $\alpha_1(\eta)$ and the stability line $\alpha_{\mathrm{AT}}(\eta)$ the solution has $H_\eta > 0$ and $1 > Q = q_1 > q_0$. Hence agents do not play pure strategies. The solution in this region depends on η.

We stress again that NE are in pure strategies, since at $\eta = 1$ for all values of α one finds $Q = 1$.

Figure 3 shows that the one-step calculation for H_1/N agrees very well with the numerical simulations and it represents a considerable improvement over the RS result[8]. Further steps of RSB, most probably infinitely many, are likely to be needed to recover exact results. However, the one-step calculation gives a rather good approximation.

[8] Note that numerical results refer to a typical NE which need not be the ground state of H_1.

4. Conclusion

Summarizing, we have analysed the solution of the minority game by means of statistical mechanics methods. Our starting point has been the study of [9, 10], where the NE of the game have been mapped onto the ground states of a disordered Hamiltonian with RSB, suggesting the existence of a very large number of NE. First, we have computed (both analytically and numerically) the number of NE, showing that there are actually exponentially many in N (number of players). Then, we probed the stability of the RS theory developed in [9]. After showing the necessity of RSB by simple entropy considerations, we calculated the instability line (AT line) using the AT method. Finally, we derived the broken-replica-symmetry solution, drawing the complete phase diagram of the model. All our results were in excellent agreement with the computer experiments.

To our knowledge, the minority game is the first example of a market game that requires the full use of spin glass theory in order to uncover its behaviour. Remarkably, many features actually observed in real markets can be recovered within the simple setup of the minority game [2–4]. Understanding real markets is one of the most challenging theoretical problems we face. This paper suggests that statistical mechanics of disordered systems may be a valuable tool in solving this problem.

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Appendix. Calculation of the AT line

The stability matrix has dimension $n(n-1) \times n(n-1)$ and is given by

$$C = \begin{pmatrix} A^{(ab,cd)} & D^{(ab,cd)} \\ D^{(ab,cd)} & B^{(ab,cd)} \end{pmatrix} \tag{32}$$

where

$$A^{(ab,cd)} = \frac{\partial^2(nF)}{\partial q_{ab}\partial q_{cd}} \qquad B^{(ab,cd)} = \frac{\partial^2(nF)}{\partial r_{ab}\partial r_{cd}} \qquad \text{and} \qquad D^{(ab,cd)} = \frac{\partial^2(nF)}{\partial q_{ab}\partial r_{cd}} \tag{33}$$

(F denotes shortly the RS free energy).

Introducing the 'perturbation' of the RS solution in the form

$$\delta q_{ab} = \zeta_{ab} \qquad \text{and} \qquad \delta r_{ab} = x\zeta_{ab} \tag{34}$$

with the condition $\sum_b \zeta_{ab} = 0$ for all a, it is possible to show that this condition is satisfied for all n by

$$\begin{aligned} &\zeta_{ab} = \zeta \qquad (a,b) \neq (1,2) \\ &\zeta_{1b} = \zeta_{2b} = \tfrac{1}{2}(3-n)\zeta \qquad b \neq 1,2 \\ &\zeta_{12} = \tfrac{1}{2}(2-n)(3-n)\zeta \qquad (a,b) = (1,2) \\ &\zeta_{aa} = 0 \qquad \forall a. \end{aligned} \tag{35}$$

The relevant eigenvalue equations (the so-called *replicon mode*) are given by

$$\begin{aligned} &\sum_{cd}(A^{(ab,cd)} + xD^{(ab,cd)})\zeta_{cd} = \lambda\zeta_{ab} \\ &\sum_{cd}(xA^{(ab,cd)} + D^{(ab,cd)})\zeta_{cd} = \lambda x\zeta_{ab}. \end{aligned} \tag{36}$$

One needs to find an expression for the matrix elements A, B and D.

There are three different types of matrix elements in the RS state, corresponding to the cases $(a,b)=(c,d)$, $a=c$ and $(a,b)\neq(c,d)$, respectively. For the $A^{(ab,cd)}$ they are $A^{(ab,ab)}$, $A^{(ab,ad)}$ and $A^{(ab,cd)}$. It is simple to show that in the RS state

$$\begin{aligned} A^{(ab,ab)} &= -\alpha\beta(E_{ab^2}+E_{aa}^2) \\ A^{(ab,ad)} &= -\alpha\beta[E_{ab}(E_{ab}+E_{aa})] \\ A^{(ab,cd)} &= -2\alpha\beta E_{ab}^2 \end{aligned} \tag{37}$$

where

$$E_{ab}=\beta q[\alpha+\beta(Q-q)]^{-2} \qquad \text{and} \qquad E_{aa}=E_{ab}+[\alpha+\beta(Q-q)]^{-1}. \tag{38}$$

For the $B^{(ab,cd)}$ we find

$$\begin{aligned} B^{(ab,ab)} &= -\alpha^2\beta^3(\langle\langle s^2\rangle^2\rangle_z-\langle\langle s\rangle^2\rangle_z^2) \\ B^{(ab,ad)} &= -\alpha^2\beta^3(\langle\langle s^2\rangle\langle s\rangle^2\rangle_z-\langle\langle s\rangle^2\rangle_z^2) \\ B^{(ab,cd)} &= -\alpha^2\beta^3(\langle\langle s\rangle^4\rangle_z-\langle\langle s\rangle^2\rangle_z^2). \end{aligned} \tag{39}$$

As for $D^{(ab,cd)}$, one finds the general result

$$D^{(ab,cd)}=\alpha\beta(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}). \tag{40}$$

It is important to notice that the relevant combinations of matrix elements appearing in the eigenvalue equations are of the form $A^{(ab,cd)}-2A^{(ab,ad)}+A^{(ab,cd)}$, and that the eigenvalues can be shown to depend only on

$$a=A^{(ab,cd)}-2A^{(ab,ad)}+A^{(ab,cd)} \qquad \text{and} \qquad b=B^{(ab,cd)}-2B^{(ab,ad)}+B^{(ab,cd)} \tag{41}$$

via the simple formula

$$\lambda_\pm=\frac{1}{2}\left(a+b\pm\sqrt{(a-b)^2+4}\right). \tag{42}$$

Putting things together and solving the eigenvalue equations, one finds that one of the eigenvalue (namely λ_-) is constant in sign (at least at low temperatures). The second one, instead, changes sign and signals the onset of RSB instability. The equation corresponding to $\lambda_+=0$ reads

$$\frac{\alpha\beta^2}{\alpha^2[1+\beta(Q-q)/\alpha]^2}\langle(\langle s^2\rangle-\langle s\rangle^2)^2\rangle_z=1. \tag{43}$$

Calculating explicitly the averages appearing in the above formula one arrives at the AT line reported in the text

$$\alpha[1-\eta(1+\beta(Q-q)/\alpha)]^2=1. \tag{44}$$

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